Abstract

The proof of: $\sum_{i=1}^{n} \frac{1}{2i-1} - \sum_{i=1}^{n} \frac{1}{2i} = \sum_{i=1}^{n} \frac{1}{n+i}$ is presented by induction on n.

1 Basis

LHS: $\sum_{i=1}^{1} \frac{1}{2i-1} - \sum_{i=1}^{1} \frac{1}{2i} = 1 - \frac{1}{2} = \frac{1}{2}$
RHS: $\sum_{i=1}^{1} \frac{1}{n+i} = \frac{1}{1+1} = \frac{1}{2}$

2 Inductive Hypothesis

Assume: $\sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=1}^{n} \frac{1}{2k} = \sum_{k=1}^{n} \frac{1}{n+k} \forall k, 1 \leq k < n.$

3 Inductive Step

Consider: $\sum_{i=1}^{n+1} \frac{1}{2i-1} - \sum_{i=1}^{n+1} \frac{1}{2i}$
$= \sum_{i=1}^{n} \frac{1}{2i-1} - \sum_{i=1}^{n} \frac{1}{2i} + \frac{1}{2(n+1)-1} - \frac{1}{2(n+1)}$
$= \sum_{i=1}^{n} \frac{1}{2i-1} - \sum_{i=1}^{n} \frac{1}{2i} + \frac{1}{2n+1} - \frac{1}{2n+2}$
$= \sum_{i=1}^{n} \frac{1}{n+i} + \frac{1}{2n+1} - \frac{1}{2n+2}$; by IH
$= \sum_{i=1}^{n} \frac{1}{n+i} + \frac{1}{2n+1} - \frac{1}{n+1}\frac{1}{2}$
$= \sum_{i=1}^{n} \frac{1}{n+i} + \frac{1}{2n+1} - \frac{1}{n+1}(1 - \frac{1}{2})$
$= \sum_{i=1}^{n} \frac{1}{n+i} + \frac{1}{2n+1} - \frac{1}{n+1} + \frac{1}{2}\frac{1}{n+1}$
$= \sum_{i=1}^{n} \frac{1}{n+i} + \frac{1}{2n+1} - \frac{1}{n+1} + \frac{1}{2n+2}$
$= \sum_{i=1}^{n+1} \frac{1}{n+1+i}$ QED

4 Explain That Last Step

Consider: $\sum_{i=1}^{n} \frac{1}{n+i}$
The first few summations look like:
$n = 1 : \frac{1}{2}$
$n = 2 : \frac{1}{3} + \frac{1}{4}$
$n = 3 : \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$
$n = 4 : \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$
Notice how the next term relates to the previous. It's basically:
$Term(n) = Term(n-1) - \frac{1}{n} + \frac{1}{2n-1} + \frac{1}{2n}$
Or:
$\sum_{i=1}^{n} \frac{1}{n+i} = \sum_{i=1}^{n-1} \frac{1}{n-1+i} - \frac{1}{n} + \frac{1}{2n-1} + \frac{1}{2n}$
Substituting n+1 for n:
$\sum_{i=1}^{n+1} \frac{1}{n+1+i} = \sum_{i=1}^{n} \frac{1}{n+i} - \frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2}$